On measures of maximal and full dimension for polynomial automorphisms of $\mathbb{C}^2$

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Abstract

For a hyperbolic polynomial automorphism of $\mathbb{C}^2$, we show the existence of a measure of maximal dimension, and identify the conditions under which a measure of full dimension exists.

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1 Introduction

Let g be a hyperbolic polynomial automorphism of $\mathbb{C}^2$. An important approach for understanding the dynamics of g is to study its invariant Borel probability measures. One key idea is to study the Hausdorff dimensions of these measures. For an invariant Borel probability measure ν, we define the Hausdorff dimension of ν by

$$\dim_H(\nu) = \inf\{\dim_H A : \nu(A) = 1\}. \tag{1.1}$$

We define the quantity $\mathrm{d}(g)$ by

$$\mathrm{d}(g) = \sup\{\dim_H(\nu)\}, \tag{1.2}$$

where the supremum is taken over all ergodic invariant Borel probability measures with positive entropy. This quantity was introduced by Denker and Urbanski [DU] in the context of rational maps on the Riemann sphere. They called it the dynamical dimension of the map.

It is easy to see that the support of each measure considered in (1.2) is contained in the Julia set J (see Section 2 for the definition). We denote by

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$M(J, g|_J)$ the set of all ergodic invariant Borel probability measures supported on J. If $\nu \in M(J, g|_J)$, then Young's formula [Y] implies that $h_\nu(g) > 0$ if and only if $\dim_H(\nu) > 0$. Thus, we could have also included measures with zero entropy while taking the supremum in (1.2).

If a measure $\nu \in M(J, g|_J)$ attains the supremum in (1.2), that is,

$$\dim_H(\nu) = \mathrm{d}(g), \tag{1.3}$$

we say that ν is a measure of maximal dimension for g.

McCluskey and Manning [MM] mentioned a heuristic argument for the existence of a measure of maximal dimension in the case of Axiom A surface diffeomorphisms. However, it is until today not known whether this argument can be extended to a rigorous proof (see the remarks after Theorem 5.1 for more details).

In this paper we study the existence of a measure of maximal dimension for hyperbolic polynomial automorphisms of $\mathbb{C}^2$. Our main result is the following.

Theorem 1.1 *Let g be a hyperbolic polynomial automorphism of $\mathbb{C}^2$. Then there exists a measure of maximal dimension for g. The set of measures of maximal dimension is finite.*

The proof of this theorem uses the theory of thermodynamic formalism. The key idea is to extract a one-parameter family of potentials and to consider the corresponding family of equilibrium measures. We show that a measure of maximal dimension necessarily belongs to this family of equilibrium measures. Furthermore, if a measure ν maximizes Hausdorff dimension among these equilibrium measures, then ν is a measure of maximal dimension.

The Hausdorff dimension of the Julia set is also an interesting dimension-theoretical feature of g. Recently, there has been made substantial progress on this subject (see [FO], [FS], [VW], [Wo1], [Wo2]). We say that $\nu \in M(J, g|_J)$ is a measure of full dimension if

$$\dim_H(\nu) = \dim_H J. \tag{1.4}$$

Friedland and Ochs [FO] studied the existence of a measure of full dimension. They proved existence for volume preserving maps. We provide an alternative proof for this result in Theorem 4.1. Moreover, we obtain in Corollary 3.6 that there exists at most one measure of full dimension. In the next theorem we consider non-volume preserving maps.

Theorem 1.2 *Let g be a non-volume preserving hyperbolic polynomial automorphism of $\mathbb{C}^2$. Assume that ν is a measure of full dimension for g. Then ν is the measure of maximal entropy of g .*

Theorem 1.2 indicates that for non-volume preserving maps the existence of a measure of full dimension seems to be a very rare phenomenon. Indeed we do not have an example of a non-volume preserving hyperbolic polynomial automorphism of $\mathbb{C}^2$ admitting a measure of full dimension.

In Section 4 we observe that there exists a dense open subset of hyperbolic parameter space for which no measure of full dimension exists. This implies that

$$\mathrm{d}(g) < \dim_H J \tag{1.5}$$

holds for these parameters.

In the last part of this paper we analyze the dependence of $\mathrm{d}(g)$ on the parameter of the mapping. More precisely, we prove the following result.

Theorem 1.3 *Let $\lambda \mapsto g_\lambda$ be a holomorphic family of hyperbolic polynomial automorphisms of $\mathbb{C}^2$ of fixed degree. Then $\lambda \mapsto \mathrm{d}(g_\lambda)$ is continuous and plurisubharmonic.*

This paper is organized as follows. In Section 2 we present the basic definitions and notations. In Section 3 we introduce elements from dimension theory for hyperbolic polynomial automorphisms of $\mathbb{C}^2$ and provide the tools for the analysis of the existence of measures of maximal and full dimension. Section 4 is devoted to the analysis of the existence of a measure of full dimension. The existence of a measure of maximal dimension is proved in Section 5. Finally, we study in Section 6 the dependence of $\mathrm{d}(g)$ on parameters.

It would be interesting to understand whether, or at least under which conditions, an uniqueness result for the measure of maximal dimension holds. A partial answer to this question is given in Corollary 3.6 where an uniqueness result is shown in the case when a measure of full dimension exists.

2 Notation and Preliminaries

Let g be a polynomial automorphism of $\mathbb{C}^2$. We can associate with g a dynamical degree d which is a conjugacy invariant (see [FM], [BS2] for details). We are interested in nontrivial dynamics which occurs if and only if $d > 1$. Friedland and Milnor showed in [FM] that every polynomial automorphism of $\mathbb{C}^2$ with nontrivial dynamics is conjugate to a mapping of the form $g = g_1 \circ ... \circ g_m$, where each g_i is a generalized Hénon mapping. This means that g_i has the form

$$g_i(z, w) = (w, P_i(w) + a_i z), \tag{2.6}$$

where P_i is a complex polynomial of degree $d_i \geq 2$ and a_i is a non-zero complex number. The dynamical degree d of g is equal to $d_1 \cdot ... \cdot d_m$ and therefore coincides with the polynomial degree of g. In this paper we assume that g is a finite composition of generalized Hénon mappings. Since dynamical properties are invariant under conjugacy, our results also hold for a general polynomial automorphism of $\mathbb{C}^2$ with nontrivial dynamics.

For g we define $K^\pm$ as the set of points in $\mathbb{C}^2$ with bounded forward/backward orbits, $K = K^+ \cap K^-$, $J^\pm = \partial K^\pm$ and $J = J^+ \cap J^-$. We refer to $J^\pm$ as the

positive/negative Julia set of g and J is the Julia set of g. The sets K and J are compact.

Note that the function $a = \det Dg$ is constant in $\mathbb{C}^2$. Therefore we can restrict our considerations to the volume decreasing case ($|a| < 1$), and to the volume preserving case ($|a| = 1$), because otherwise we can consider g^{-1}. This will be a standing assumption in this paper. We note that for g we have $a = a_1 \cdot ... \cdot a_m$.

As pointed out in the introduction we assume in this paper that g is a hyperbolic mapping. This means that there exists a continuous invariant splitting $T_J\mathbb{C}^2 = E^u \oplus E^s$ such that $Dg|_{E^u}$ is uniformly expanding and $Dg|_{E^s}$ is uniformly contracting. Hyperbolicity implies that we can associate with each point $p \in J$ its local unstable/stable manifold $W_\epsilon^{u/s}(p)$. Moreover g is an Axiom A diffeomorphism (see [BS1] for more details).

3 Elements from dimension theory

In this section we introduce elements from dimension theory for hyperbolic polynomial automorphisms of $\mathbb{C}^2$ and provide the tools for the analysis of measures of maximal and full dimension.

We start by introducing Lyapunov exponents. Let $\nu \in M(J, g|_J)$. By the multiplicative ergodic Theorem of Oseledec, there are Lyapunov exponents $\lambda_\nu^{(1)} \leq \lambda_\nu^{(2)}$ with respect to ν (see e.g. [KH]). The Julia set J is a hyperbolic set of index 1 (see [BS1]). This implies

$$\lambda_\nu^{(1)} < 0 < \lambda_\nu^{(2)}. \tag{3.7}$$

In particular, ν is a hyperbolic measure. We define the quantity

$$\Lambda(\nu) = \lim_{n\to\infty} \frac{1}{n} \int \log ||Dg^n|| d\nu. \tag{3.8}$$

Similar as it was done for the measure of maximal entropy in [BS3], the positive Lyapunov exponent $\lambda_\nu^{(2)}$ coincides with $\Lambda(\nu)$. Since g has constant jacobian determinant a, the negative Lyapunov exponent $\lambda_\nu^{(1)}$ is given by $-\Lambda(\nu) + \log |a|$. In [Wo1] we derived the formula

$$\Lambda(\nu) = \int \log ||Dg|_{E^u}|| d\nu. \tag{3.9}$$

By Young's formula [Y], we have for all $\nu \in M(J, g|_J)$ that

$$\dim_H(\nu) = \frac{h_\nu(g)}{\Lambda(\nu)} + \frac{h_\nu(g)}{\Lambda(\nu) - \log |a|}. \tag{3.10}$$

Here $h_\nu(g)$ denotes the measure theoretic entropy of g with respect to ν.

Next we introduce topological pressure. Let $C(J, \mathbf{R})$ denote the Banach space of all continuous functions from J to $\mathbf{R}$. The topological pressure of $g|_J$,

denoted by $P = P(g|_J, .)$, is a mapping from $C(J, \mathbf{R})$ to $\mathbf{R}$ (see [Wa] for the definition). We have $P(g|_J, 0) = h_{top}(g) = \log d$, where $h_{top}(g)$ denotes the topological entropy of g and d is the polynomial degree of g. The variational principle provides the formula

$$P(g|_J, \varphi) = \sup_{\nu \in M(J, g|_J)} \left(h_\nu(g) + \int_J \varphi d\nu \right). \tag{3.11}$$

If a measure $\nu_\varphi \in M(J, g|_J)$ achieves the supremum in equation (3.11), that is,

$$P(g|_J, \varphi) = h_{\nu_\varphi}(g) + \int_J \varphi d\nu_\varphi, \tag{3.12}$$

it is called equilibrium measure of the potential φ.

The topological pressure has the following properties (see [R]).

i) The topological pressure is a convex function.

ii) If φ is a strictly negative function, then the function $t \mapsto P(g|_J, t\varphi)$ is strictly decreasing.

iii) The topological pressure is a real analytic function on the subspace of Hölder continuous functions, that is, when $\alpha > 0$ is fixed, then $P(g|_J, .)|_{C^\alpha(J, \mathbf{R})}$ is a real analytic function. Note that C^α can not be replaced by C^0.

iv) If $\alpha > 0$ and $\varphi \in C^\alpha(J, \mathbf{R})$, then there exists a uniquely defined equilibrium measure $\nu_\varphi \in M(J, g|_J)$ of the potential φ. Furthermore we have for all $\varphi, \psi \in C^\alpha(J, \mathbf{R})$

$$\left. \frac{d}{dt} \right|_{t=0} P(g|_J, \varphi + t\psi) = \int_\Lambda \psi d\nu_\varphi. \tag{3.13}$$

We now introduce potentials which are related to Lyapunov exponents. We define

$$\phi^{u/s} : J \to \mathbf{R}, \qquad p \mapsto \log ||Dg(p)|_{E_p^{u/s}}|| \tag{3.14}$$

and the unstable/stable pressure function

$$P^{u/s} : \mathbf{R} \to \mathbf{R}, \qquad t \mapsto P(g|_J, \mp t\phi^{u/s}). \tag{3.15}$$

Since $\phi^{u/s}$ is Hölder continuous (see [B]), we may follow from property iii) of the topological pressure that $P^{u/s}$ is real analytic. Property iv) of the topological pressure implies that there exists a uniquely defined equilibrium measure $\nu_{\mp t\phi^{u/s}} \in M(J, g|_J)$ of the potential $\mp t\phi^{u/s}$.

We will need the following result about the relation between the unstable and stable pressure function.

Proposition 3.1 ([Wo2]) $P^u(t) = P^s(t) - t \log |a|$.

Lemma 3.2 $\nu_{-t\phi^u} = \nu_{t\phi^s}$

Proof. Let $t \geq 0$. Then

$$
\begin{aligned}
P^s(t) &= P^u(t) + t\log|a| \\
&= h_{\nu_{-t\phi^u}}(g) + t\left(-\int \log ||Dg|_{E^u}||d\nu_{-t\phi^u} + \log|a|\right) \\
&= h_{\nu_{-t\phi^u}}(g) + t\left(\lim_{n\to\infty}\frac{1}{n}\int -\log||Dg^n|_{E^u}|| + \log|a^n|d\nu_{-t\phi^u}\right) \\
&= h_{\nu_{-t\phi^u}}(g) + t\left(\lim_{n\to\infty}\frac{1}{n}\int \log||Dg^n|_{E^s}||d\nu_{-t\phi^u}\right) \\
&= h_{\nu_{-t\phi^u}}(g) + t\int \log||Dg|_{E^s}||d\nu_{-t\phi^u} \\
&= h_{\nu_{-t\phi^u}}(g) + t\int \phi^s d\nu_{-t\phi^u}.
\end{aligned} \tag{3.16}
$$

The result follows from the uniqueness of the equilibrium measure of the potential $t\phi^s$. $\square$

We will use in the remainder of this paper the notation $\nu_t = \nu_{\mp t\phi^{u/s}}$. This notation is justified by Lemma 3.2. We also write $\Lambda(t) = \Lambda(\nu_t)$ and $h(t) = h_{\nu_t}(g)$, and consider Λ and h as real-valued functions of t. Equations (3.9), (3.12) imply

$$P^u(t) = h(t) - t\Lambda(t). \tag{3.17}$$

Therefore, Proposition 3.1 implies

$$P^s(t) = h(t) - t(\Lambda(t) - \log|a|). \tag{3.18}$$

Proposition 3.3 *Λ and h are real analytic. Furthermore*

$$\frac{d\Lambda}{dt} \leq 0. \tag{3.19}$$

If Λ is not constant, then every zero of the derivative of Λ is isolated.

Proof. Let $t_0 \geq 0$. We define potentials $\varphi = -t_0\phi^u, \psi = -\phi^u$. Therefore application of equations (3.9) and (3.13) imply

$$\frac{dP^u}{dt}(t_0) = -\Lambda(\nu_{t_0}) = -\Lambda(t_0). \tag{3.20}$$

Since P^u is real analytic, we obtain that Λ is also real analytic. We conclude from (3.17) that h is also real analytic. The convexity of P^u implies

$$\frac{d^2P^u}{dt^2} \geq 0, \tag{3.21}$$

hence

$$\frac{d\Lambda}{dt} \leq 0. \tag{3.22}$$

Finally, if Λ is not constant, then the uniqueness theorem for real analytic functions, applied to the derivative of Λ, implies that all zeros of the derivative of Λ are isolated. □

Corollary 3.4 *Λ is either constant or strictly decreasing.*

Proof. The result follows immediately from Proposition 3.3. □

Of particular interest for the analysis of the existence of measures of maximal and full dimension are the Hausdorff dimensions of the measures ν_t. We use the notation $\Delta(t) = \dim_H(\nu_t)$. Equation (3.10) yields

$$\Delta(t) = \frac{h(t)}{\Lambda(t)} + \frac{h(t)}{\Lambda(t) - \log|a|}. \tag{3.23}$$

Thus, Δ is also a real analytic function. Equations (3.17), (3.18) and Proposition 3.1 imply

$$\Delta(t) = 2t + \frac{P^u(t)}{\Lambda(t)} + \frac{P^u(t) + t\log|a|}{\Lambda(t) - \log|a|}. \tag{3.24}$$

From an elementary calculation we obtain the following formula for the derivative of Δ.

$$\frac{d\Delta}{dt}(t_0) = -\frac{\frac{d\Lambda}{dt}(t_0)\left[P^u(t_0)(\Lambda(t_0) - \log|a|)^2 + (P^u(t_0) + t_0\log|a|)\Lambda(t_0)^2\right]}{\Lambda(t_0)^2(\Lambda(t_0) - \log|a|)^2}. \tag{3.25}$$

Finally we consider the Hausdorff dimension of the Julia set J. The following result due to Verjovsky and Wu provides a formula for the Hausdorff dimension of the unstable/stable slice in terms of the zeros of the pressure functions.

Theorem 3.5 ([VW]) *Let $p \in J$. Then $t^{u/s} = \dim_H W_\epsilon^{u/s}(p) \cap J$ does not depend on $p \in J$. Furthermore $t^{u/s}$ is given by the unique solution of*

$$P^{u/s}(t) = 0. \tag{3.26}$$

Equation (3.26) is called Bowen-Ruelle formula. We refer to $t^{u/s}$ as the Hausdorff dimension of the unstable/stable slice.

In [Wo1] we proved the formula

$$\dim_H J = t^u + t^s = \sup_{\nu \in M(J,g|_J)} \left(\frac{h_\nu(g)}{\Lambda(\nu)}\right) + \sup_{\nu \in M(J,g|_J)} \left(\frac{h_\nu(g)}{\Lambda(\nu) - \log|a|}\right), \tag{3.27}$$

where each of the suprema on the right-hand side of the equation is uniquely attained by the measures ν_{t^u} and ν_{t^s} respectively. Hence

$$\dim_H J = \frac{h(t^u)}{\Lambda(t^u)} + \frac{h(t^s)}{\Lambda(t^s) - \log|a|}. \tag{3.28}$$

Equation (3.28) and the uniqueness of the measures ν_{t^u}, ν_{t^s} in equation (3.27) imply that, if there exists a measure of full dimension, then it already coincides with ν_{t^u} and ν_{t^s}. Thus, we have the following result.

Corollary 3.6 *Assume m is a measure of full dimension for g, then $m = \nu_{t^u} = \nu_{t^s}$. In particular, there exists at most one measure of full dimension.*

4 Measures of full dimension

In this section we identify the conditions under which a measure of full dimension exists. More precisely, we show that a measure of full dimension exists if and only if g is either volume preserving, or g is volume decreasing and the measure of maximal entropy is a measure of full dimension.

We start with the volume preserving case.

Theorem 4.1 *Let g be volume preserving. Then $t^u = t^s$, and ν_{t^u} is a measure of full dimension for g.*

Proof. We have $|a| = 1$, therefore Proposition 3.1 implies $P^u = P^s$. Thus, Theorem 3.5 yields $t^u = t^s$. Therefore, by equations (3.10), (3.28), we conclude that $\dim_H(\nu_{t^u}) = \dim_H J$, which implies that ν_{t^u} is a measure of full dimension. □

Remark. As noted in the introduction, in the volume preserving case the existence of a measure of full dimension was already shown by Friedland and Ochs [FO]. They proved that the existence of a measure of full dimension follows from the fact that $|\det Dg^n(p)| = 1$ holds for every periodic point p with period n. They also observed that in this case the measure of full dimension is equivalent to the t-dimensional Hausdorff measure, where t is the Hausdorff dimension of J.

We now consider the volume decreasing case. The following theorem is the main result of this section.

Theorem 4.2 *Assume g is volume decreasing. Then the following are equivalent.*

i) g admits a measure of full dimension.

ii) The unstable pressure function P^u is affine.

iii) The stable pressure function P^s is affine.

iv) The measure of maximal entropy is a measure of full dimension for g.

Proof.
$ii) \Leftrightarrow iii)$ follows from Proposition 3.1.
$i) \Rightarrow ii)$ Let us assume that g admits a measure of full dimension. Corollary 3.6 implies that $\nu_{t^u} = \nu_{t^s}$ is the measure of full dimension. Since $t^s < t^u$ (see [Wo2]), Corollary 3.4 implies that P^u has constant derivative in $[t^s, t^u]$. Therefore, since P^u is real analytic, we may conclude that P^u is affine.
$ii) + iii) \Rightarrow iv)$ Let μ denote the measure of maximal entropy. The topological entropy of $g|_J$ is equal to $\log d$ (see [BS3]). Thus $P^u(0) = P^s(0) = \log d$. Equation 3.20 and Proposition 3.1 imply

$$\frac{dP^u}{dt}(0) = -\Lambda(\mu) \tag{4.29}$$

$$\frac{dP^s}{dt}(0) = -\Lambda(\mu) + \log|a|. \tag{4.30}$$

Since P^u and P^s are affine, Theorem 3.5 and equation (3.27) imply

$$\dim_H J = \frac{\log d}{\Lambda(\mu)} + \frac{\log d}{\Lambda(\mu) - \log|a|}. \tag{4.31}$$

But by Young's formula (3.10), the right-hand side of equality (4.31) is equal to $\dim_H(\mu)$. Thus, μ is a measure of full dimension.
$iv) \Rightarrow i)$ trivial. □

Corollary 4.3 *Assume g is volume decreasing. If there exists a measure of full dimension for g, then J is either a Cantor set or connected.*

Proof. Let μ denote the measure of maximal entropy. It is shown in [BS3] that $\Lambda(\mu) \geq \log d$. The Julia set J is connected, if and only if $\Lambda(\mu) = \log d$ (see [BS6]). Therefore, if J is not connected, then $\Lambda(\mu) > \log d$. Thus, equation (3.27) implies $t^u, t^s < 1$. Since J has a local product structure (see [BS1]), we may conclude that J is a Cantor set. □

Let us assume that g is volume decreasing and let μ denote the measure of maximal entropy of g. We assume that g admits a measure of full dimension; thus, by Theorem 4.2, μ is the measure of full dimension for g. Let $\mathcal{S}$ denote the set of all saddle points of g. For $p \in \mathcal{S}$ with period n we denote by $\lambda^{u/s}(p)$ the eigenvalues of $Dg^n(p)$, where $|\lambda^s(p)| < 1 < |\lambda^u(p)|$. Then, we may conclude by Theorem 4.2 and Proposition 4.5 of [B] that

$$\log|\lambda^u(p)| = n\Lambda(\mu) \tag{4.32}$$

holds for all $p \in \mathcal{S}$. Using a perturbation argument, it is not to hard to see that we can find arbitrary close to g a hyperbolic polynomial automorphism of $\mathbb{C}^2$ for which (4.32) does not hold, and which therefore does not admit a

measure of full dimension. Here we mean close with respect to the topology on hyperbolic parameter space induced by the parameter of the mapping (see [Wo1] for details). On the other hand, it is obvious that the set of parameters having no measure of full dimension is an open subset of hyperbolic parameter space. Thus, there exists a dense open subset of parameters admitting no measure of full dimension. We leave the details to the reader.

Remarks. It is a well-known fact that for hyperbolic and parabolic rational maps on the Riemann sphere the Hausdorff dimension of the Julia set can be represented in terms of the Bowen-Ruelle formula (see for instance [U] and the references therein). Therefore, for these maps the existence of a measure of full dimension follows as a consequence of the variational principle. For results concerning the existence of measures of full dimension for other maps see [GP] and the references therein.

5 Measures of maximal dimension

In this section we consider the case when g does not admit a measure of full dimension. Thus, by the results of the previous section, we may assume that g is volume decreasing and $P^{u/s}$ is not affine. This will be a standing assumption in this section. The following theorem is the main result of this paper.

Theorem 5.1 *There exists a measure of maximal dimension for g. If m is a measure of maximal dimension for g, then there exists $t^s < t < t^u$ such that m is the equilibrium measure of the potential $-t\phi^u$, that is $m = \nu_t$.*

Proof. Since g is volume decreasing, we have $t^s < t^u$ (see [Wo2]).
Assertion 1. There exists $\epsilon > 0$ such that Δ is strictly increasing on $[0, t^s + \epsilon)$ and strictly decreasing on $(t^u - \epsilon, \infty)$.

Proof of Assertion 1. Theorem 3.5 and the fact that $P^{u/s}$ is a strictly decreasing function [property ii) of the topological pressure] imply that $P^s(t) > 0$ for all $t \in [0, t^s)$. Analogously we have $P^u(t) > 0$ for all $t \in [0, t^u)$. We conclude from Proposition 3.3, equation (3.25) and an elementary continuity argument that there exists $\epsilon > 0$ such that

$$\frac{d\Delta}{dt} \geq 0 \tag{5.33}$$

in $[0, t^s + \epsilon)$, and all zeros of the derivative of Δ in $[0, t^s + \epsilon)$ are isolated. Therefore, Δ is strictly increasing in $[0, t^s + \epsilon)$. A similar argumentation shows that there exists $\epsilon > 0$ such that Δ is strictly decreasing in $(t^u - \epsilon, \infty)$.
Assertion 1 implies that there exists $t^* \in [t^s + \epsilon, t^u - \epsilon]$ such that

$$\dim_H(\nu_{t^*}) = \sup_{t \geq 0} \Delta(t). \tag{5.34}$$

Assertion 2. The measure ν_{t^*} is a measure of maximal dimension.

Proof of Assertion 2. Let $(m_k)_{k\in\mathbf{N}}$ be a sequence in $M(J, g|_J)$ such that

$$\lim_{k\to\infty} \dim_H(m_k) = \mathrm{d}(g). \tag{5.35}$$

By Assertion 1, we may assume without loss of generality that $\dim_H(\nu_0) = \Delta(0) < \dim_H(m_k)$ for all $k \in \mathbf{N}$. Since ν_0 is the unique measure of maximal entropy, we may conclude by Young's formula (3.10) that

$$\Lambda(\nu_0) > \Lambda(m_k) \tag{5.36}$$

for all $k \in \mathbf{N}$. Again by Assertion 1, we may assume without loss of generality that $\dim_H(\nu_{t^u}) = \Delta(t^u) < \dim_H(m_k)$ for all $k \in \mathbf{N}$. Equation (3.27) implies

$$\frac{h_{m_k}(g)}{\Lambda(m_k)} < \frac{h(t^u)}{\Lambda(t^u)} \tag{5.37}$$

for all $k \in \mathbf{N}$. Therefore, Young's formula (3.10) implies

$$\frac{h_{m_k}(g)}{\Lambda(m_k) - \log|a|} > \frac{h(t^u)}{\Lambda(t^u) - \log|a|} \tag{5.38}$$

for all $k \in \mathbf{N}$. Equations (5.37), (5.38) imply $h_{m_k}(g) > h(t^u)$, and therefore again by equation (5.37) we obtain

$$\Lambda(m_k) > \Lambda(t^u) \tag{5.39}$$

for all $k \in \mathbf{N}$. Since Λ is continuous, equations (5.36), (5.39) imply that for all $k \in \mathbf{N}$ there exists $t_k \in (0, t^u)$ such that

$$\Lambda(m_k) = \Lambda(t_k). \tag{5.40}$$

Thus, the variational principle (3.11) implies

$$h_{m_k}(g) \leq h(t_k), \tag{5.41}$$

hence

$$\dim_H(m_k) \leq \Delta(t_k) \tag{5.42}$$

for all $k \in \mathbf{N}$. This implies

$$\dim_H(m_k) \leq \dim_H(\nu_{t^*}) \tag{5.43}$$

for all $k \in \mathbf{N}$. We conclude that ν_{t^*} is a measure of maximal dimension.
To complete the proof of the theorem we have to show the following.
Assertion 3. For every measure m of maximal dimension there exists $t^s < t < t^u$ such that m is the equilibrium measure of the potential $-t\phi^u$.

Proof of Assertion 3. Let m be a measure of maximal dimension. We apply to m (instead of m_k) the same argumentation as in the proof of Assertion 2. This implies that there exists $t \in (0, t^u)$ such that $\Lambda(m) = \Lambda(t)$. Since

$\dim_H(m) \geq \Delta(t)$, we may follow by equation (3.10) that $h_m(g) \geq h(t)$. On the other hand, since ν_t is the equilibrium measure of the potential $-t\phi^u$, we may conclude by (3.11), (3.12) that $h_m(g) \leq h(t)$. Hence $h_m(g) = h(t)$. Therefore, the uniqueness of the equilibrium measure of the potential $-t\phi^u$ implies $m = \nu_t$. Finally, Assertion 1 implies that $t \in (t^s, t^u)$. This completes the proof. $\Box$

Remarks. The following heuristic argument was mentioned by McCluskey and Manning [MM] to state the existence of a measure of maximal dimension in the case of C^2 axiom A diffeomorphisms of real surfaces. Since the entropy map is upper semi-continuous it can be shown that the map $\nu \mapsto dim_H(\nu)$, defined on the set of all ergodic invariant measures, is also upper semi-continuous. It is now suggested in [MM] that this implies the existence of a measure of maximal dimension. To make this argument rigorous we need to show that there exists a sequence of ergodic invariant measures m_k with $\dim_H(m_k) \to \mathrm{d}(g)$ having an ergodic weak* limit. Whether this holds is not clear since the set of all ergodic invariant measures is not closed with respect to the weak* topology. The latter follows for instance from Proposition 21.9 in [DGS].

We obtain a formula for $\mathrm{d}(g)$.

Corollary 5.2 *Let $t > 0$ such that ν_t is a measure of maximal dimension. Then*

$$\mathrm{d}(g) = 2t + \frac{P^u(t) \log |a|}{\Lambda(t)^2}. \tag{5.44}$$

Proof. By Proposition 3.3, equation (3.25) and Theorem 5.1, a necessary condition for ν_t being a measure of maximal dimension is

$$P^u(t)(\Lambda(t) - \log |a|)^2 + (P^u(t) + t \log |a|)\Lambda(t)^2 = 0. \tag{5.45}$$

Therefore, the result follows from equation (3.24). $\Box$

Corollary 5.3 *The set of all measures of maximal dimension is finite.*

Proof. The function Δ is a non-constant real analytic function on $[0, \infty)$. Therefore, it follows from the uniqueness theorem for real analytic functions that Δ has only finitely many maxima in $[t^s, t^u]$. The result follows from Theorem 5.1. $\Box$

Corollary 5.4 $\mathrm{d}(g) < \dim_H J$

Proof. Assume $\mathrm{d}(g) = \dim_H J$. By Theorem 5.1, there exists a measure of maximal dimension. Therefore, this measure is a measure of full dimension. But this contradicts the standing assumption of this section that g does not admit a measure of full dimension. $\Box$

Corollary 5.5 *Every measure ν of maximal dimension is Bernoulli.*

Proof. Since $g|_J$ is topological mixing (see [BS1]), the result follows from the fact that ν is an equilibrium measure of a Hölder continuous potential (see [B], Thm. 4.1). $\square$

6 Dependence on Parameters

Let A denote an open subset of $\mathbb{C}^k$ and let $(g_\lambda)_{\lambda \in A}$ be a holomorphic family of hyperbolic mappings of the form $g_\lambda = g_{\lambda_1} \circ \cdots \circ g_{\lambda_m}$, where each of the mappings g_{λ_i} is a generalized Hénon mapping of fixed degree d_i (see [Wo1] for more details). We denote by J_λ the Julia set, by a_λ the Jacobian determinant, and by $P_\lambda^{u/s}$ the unstable/stable pressure function of g_λ. We also write $\Delta_\lambda(t)$ instead of $\Delta(t)$. First, we show that $\mathrm{d}(g)$ depends continuously on the parameter of the mapping.

Theorem 6.1 *The function $\lambda \mapsto \mathrm{d}(g_\lambda)$ is continuous in A.*

Proof. Let $\lambda_0 \in A$. The result of [VW] implies that there exist $\epsilon > 0$ and a real analytic function

$$\mathcal{P} : B(\lambda_0, \epsilon) \times [0, \infty) \to \mathbf{R}, \tag{6.46}$$

such that $\mathcal{P}(\lambda, .) = P_\lambda^u$ for all $\lambda \in B(\lambda_0, \epsilon)$. Therefore equations (3.20), (3.24) imply that

$$\mathcal{D} : B(\lambda_0, \epsilon) \times [0, \infty) \to \mathbf{R}, \quad (\lambda, t) \mapsto \Delta_\lambda(t) \tag{6.47}$$

is also a real analytic function. Now we may conclude by Corollary 3.6 and Theorem 5.1 that

$$\mathrm{d}(g_\lambda) = \max_{t \in [0,2]} \mathcal{D}(\lambda, t). \tag{6.48}$$

The result follows by an elementary continuity argument. $\square$

Remark. McCluskey and Manning [MM] considered C^2 Axiom A diffeomorphisms of real surfaces. They showed that for these mappings $\mathrm{d}(g)$ depends continuously on the mapping with respect to C^2 topology.

Finally, we show that $\mathrm{d}(g)$ depends plurisubharmonically on the parameter of mapping.

Theorem 6.2 *The function $\lambda \mapsto \mathrm{d}(g_\lambda)$ is plurisubharmonic in A.*

Proof. Let $g_0 \in A$ and let L be a complex line in $\mathbb{C}^k$ containing g_0. Then there exists a holomorphic family $(g_\lambda)_{\lambda \in D}$, where D is a disk with center 0 in $\mathbb{C}$ such that $\{g_\lambda : \lambda \in D\}$ is a neighborhood of g_0 in $L \cap A$. If the radius of

D is small enough, then there exists a family $(\kappa_\lambda)_{\lambda \in D}$, where each κ_λ is the uniquely defined conjugacy between $g_0|_{J_0}$ and $g_\lambda|_{J_\lambda}$. Therefore, $T_\lambda = (\kappa_\lambda)_*$ defines a family of bijections from $M(J_0, g_0|_{J_0})$ to $M(J_\lambda, g_\lambda|_{J_\lambda})$. Moreover we have $h_\nu(g_0) = h_{T_\lambda(\nu)}(g_\lambda)$ for all $\nu \in M(J_0, g_0|_{J_0})$ and all $\lambda \in D$ (see [Wo1] for the details). In [Wo1] we showed that if $\nu \in M(J_0, g_0|_{J_0})$ is fixed, then $\lambda \mapsto \Lambda(T_\lambda(\nu))$ is a harmonic function in D. We conclude by Young's formula (3.10) that

$$\mathrm{d}(g_\lambda) = \sup_{\nu \in M(J_0, g_0|_{J_0})} \left(\frac{h_\nu(g_0)}{\Lambda(T_\lambda(\nu))} + \frac{h_\nu(g_0)}{\Lambda(T_\lambda(\nu)) - \log|a_\lambda|} \right). \tag{6.49}$$

The functions $\lambda \mapsto \Lambda(T_\lambda(\nu))$, $\lambda \mapsto \Lambda(T_\lambda(\nu)) - \log|a_\lambda|$ are harmonic in D. Note that $x \mapsto x^{-1}$ is a convex function on $\mathbf{R}^+$. This implies that the functions $\lambda \mapsto \Lambda(T_\lambda(\nu))^{-1}$, $\lambda \mapsto (\Lambda(T_\lambda(\nu)) - \log|a_\lambda|)^{-1}$ are subharmonic in D. The continuous function $\lambda \mapsto \mathrm{d}(g_\lambda)$ is therefore given by the supremum over a family of subharmonic functions. We conclude that the function $\lambda \mapsto \mathrm{d}(g_\lambda)$ is subharmonic in D. This completes the proof. $\square$

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